TRACE ESTIMATES FOR UNIMODAL LÉVY PROCESSES

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Abstract. We give two-term small-time approximation for the trace of the Dirichlet heat kernel of bounded smooth domain for unimodal Lévy processes satisfying the weak scaling conditions.

1. Introduction

A two-term small-time uniform approximation for the trace of the transition density of the Wiener process killed off bounded R-smooth domain $D \subset \mathbb{R}^d$, i.e. the classical Dirichlet heat kernel, was obtained by van den Berg [16]. The first term of the approximation is proportional to the domain's volume $|D|$ and the second–to the surface measure $|\partial D|$ of the boundary, with explicit coefficient depending on time. Asymptotic non-uniform expansions of the trace of the heat kernel were given earlier in [11], see the discussion in [16].

Bañuelos and Kulczycki [1] obtained a uniform two-term approximation for the isotropic α-stable Lévy processes. The closely related case of the relativistic α-stable Lévy processes was resolved by Bañuelos, Mijena and Nane [3]. A similar two-term approximation for Lipschitz domains was given for the Wiener process by Brown [8], and for the isotropic α-stable Lévy processes–by Bañuelos, Kulczycki and Siudeja [2]. Park and Song [12] obtained a two-term small-time approximation of the trace for the relativistic α-stable Lévy processes on Lipschitz domains, and gave an explicit power expansion of the first term.

In this work we investigate those Lévy processes X_t in $\mathbb{R}^d$, where $d \geq 2$, which are unimodal and satisfy the so-called weak lower and upper scaling conditions, denoted WLSC and WUSC respectively, of orders strictly between 0 and 2 (see Section 2 for details). The isotropic stable and relativistic Lévy processes are included as special cases but at present the orders of the lower and upper scalings may differ. For bounded R-smooth open sets $D \subset \mathbb{R}^d$ (also called $C^{1,1}$ open sets in the literature) our main result gives a two-term small-time approximation of the trace of the corresponding Dirichlet heat kernel. For instance we resolve sums of independent isotropic stable Lévy processes with different indexes.

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In what follows we let ψ be the Lévy-Khintchine exponent and $p_t(x)$ be the transition density of X_t. We consider

$$\tau_D = \{t > 0 : X_t \notin D\},$$

the first time that X_t exits D. For $t > 0$ and $x, y \in \mathbb{R}^d$, we define the heat remainder

$$r_D(t,x,y) = \mathbb{E}^x \left[\tau_D < t, p_{t-\tau_D}(X(\tau_D) - y)\right]. \tag{1}$$

The Dirichlet heat kernel for X_t is given by the Hunt formula:

$$p_D(t,x,y) = p_t(y-x) - r_D(t,x,y), \tag{2}$$

and the trace of X_t on D is

$$\mathrm{tr}(t,D) = \int p_D(t,x,x)dx, \qquad t > 0. \tag{3}$$

We denote $\mathbb{H} = \{(x_1,\ldots,x_d) \in \mathbb{R}^d : x_1 > 0\}$, a half-space, and for $t > 0$ we let

$$C_{\mathbb{H}}(t) = \int_0^\infty r_{\mathbb{H}}(t,(q,0,\cdots,0),(q,0,\cdots,0))dq.$$

For instance, $C_{\mathbb{H}}(t) = ct^{-d/\alpha+1/\alpha}$ for the isotropic α-stable Lévy process [1]. Here is our main result (a stronger statement is given as Theorem 3.1 in Section 3).

Theorem 1.1. *If bounded open set $D \subset \mathbb{R}^d$ is R-smooth,* WLSC *and* WUSC *hold for ψ, and $t \to 0$, then $\mathrm{tr}(t,D)$ equals $p_t(0)|D| - C_{\mathbb{H}}(t)|\partial D|$ plus lower order terms.*

Heuristically, if $x \in D$ and $t > 0$ is small, then $r_D(t,x,x)$ is small and so $p_D(t,x,x)$ is close to $p_{\mathbb{R}^d}(t,x,x) = p_t(0)$. Therefore the first approximation to $\mathrm{tr}(t,D)$ is $p_t(0)|D|$. The second term in Theorem 1.1, $C_{\mathbb{H}}(t)|\partial D|$, approximates $\int_D r_D(t,x,x)dx$. As we shall see, $r_D(t,x,x)$ depends primarily on the distance of x from ∂D. It is here that the R-smoothness of D plays a role by allowing for an asymptotic coefficient independent of D, that is $C_{\mathbb{H}}(t)$. In view of the definition of $C_{\mathbb{H}}(t)$, the appearance of $|\partial D|$ in the second term of the approximation of the trace is natural.

In some cases, including the relativistic stable Lévy process, explicit expansions of $p_t(0)$ can be given [12, Lemma 3.2]. In more general situations $p_t(0)$, $C_{\mathbb{H}}(t)$ and the bounds for the error terms cannot be entirely explicit but Lemma 2.7 and Theorem 3.1 below provide a satisfactory formulation.

Technically we only need to estimate $\int_D r_D(t,x,x)dx$ to prove Theorem 1.1. In this connection we note that sharp global estimates for $p_D(t,x,y)$ were recently obtained by Bogdan, Grzywny and Ryznar [6], but these estimates do not easily translate into sharp estimates of $r_D(t,x,y)$. Namely, if $p_D(t,x,y)$ is only known to be proportional to $p_t(y-x)$, then essential further work is needed to accurately estimate $r_D(t,x,y)$.

The paper is composed as follows. In Section 2 we give preliminaries on unimodal Lévy processes with scaling, their heat kernel, Green function and Poisson kernel for R-smooth open sets. In Section 3 we prove Theorem 3.1, a stronger and more detailed variant of Theorem 1.1. The most technical step of the proof of Theorem 3.1 is given separately in Section 4.

We remark in passing that the trace can also be studied and interpreted within the spectral theory of the corresponding semigroup given by the integral kernel p_D [1]. In view toward further research we note that sharp pointwise estimates of $r_D(t,x,y)$ complementing [6] would be of considerable interest. We also note that two-term approximations of the trace of the heat kernel of general unimodal Lévy processes are open for Lipschitz domains.

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2. Preliminaries

2.1. **Unimodality.** A Borel measure on $\mathbb{R}^d$ is called isotropic unimodal, in short: unimodal, if on $\mathbb{R}^d \setminus \{0\}$ it is absolutely continuous with respect to the Lebesgue measure and has a radially nonincreasing, in particular rotationally invariant, or isotropic density function. Recall that Lévy measure is an arbitrary Borel measure concentrated on $\mathbb{R}^d \setminus \{0\}$ and such that

$$\int_{\mathbb{R}^d} \left(|x|^2 \wedge 1\right) \nu(dx) < \infty.$$

In what follows we assume that ν is a unimodal Lévy measure and define

$$\psi(\xi) = \int_{\mathbb{R}^d} \left(1 - \cos\langle \xi, x\rangle\right) \nu(dx), \qquad \xi \in \mathbb{R}^d, \tag{4}$$

the Lévy-Khintchine exponent. It is a radial function, and we often let $\psi(r) = \psi(\xi)$, where $\xi \in \mathbb{R}^d$ and $r = |\xi| \geq 0$. The same convention applies to all radial functions. The (radially nonincreasing) density function of the unimodal Lévy measure ν will also be denoted by ν, so $\nu(dx) = \nu(x)dx$ and $\nu(x) = \nu(|x|)$. We point out that for $\lambda \geq 1$ and $r \geq 0$, $\psi(\lambda r) \geq \pi^{-2}\psi(r)$ and $\psi(\lambda r) \leq \pi^{-2}\lambda^2\psi(r)$ [5, Section 4]. More restrictive inequalities of this type define what are called the weak scaling conditions, see Section 2.2.

We consider the pure-jump Lévy process $X = (X_t,\, t \geq 0)$ on $\mathbb{R}^d$ [13], in short: X_t, determined by the Lévy-Khintchine formula

$$\mathbb{E}\, e^{i\langle \xi, X_t\rangle} = e^{-t\psi(\xi)} = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p_t(dx).$$

The process is (isotropic) unimodal, meaning that all its one-dimensional distributions $p_t(dx)$ are (isotropic) unimodal; in fact the unimodality of ν is also necessary for the unimodality of X_t [17]. In what follows we always assume that ψ is unbounded, equivalently that $\nu(\mathbb{R}^d) = \infty$. In other words X_t below is not a compound Poisson process. Clearly, $\psi(0) = 0$ and $\psi(u) > 0$ for $u > 0$. By [6, Lemma 1.1], $p_t(dx)$ have bounded, in fact smooth density functions $p_t(x)$ for all $t > 0$ if and only if the following Hartman-Wintner condition holds,

$$\lim_{|\xi|\to\infty} \psi(\xi)/\ln|\xi| = \infty. \tag{5}$$

Let V be the renewal function of the corresponding ladder-height process of the first coordinate of X_t. Namely we consider $X_t^{(1)}$, the first coordinate process of X_t, its running maximum $M_t := \sup_{0\le s\le t} X_s^{(1)}$ and the local time L_t of $M_t - X_t^{(1)}$ at 0 so normalized that its inverse function L_t^{-1} is a standard 1/2-stable subordinator. The resulting ladder-height process $\eta(t) := X^{(1)}(L_t^{-1})$ is a subordinator with the Laplace exponent

$$\kappa(u) = -\log \mathbb{E}e^{-u\eta(1)} = \exp\left\{\frac{1}{\pi}\int_0^\infty \frac{\log\psi(u\zeta)}{1+\zeta^2}\,d\zeta\right\}, \quad u \ge 0,$$

and $V(x)$ is defined as the accumulated potential of η:

$$V(x) = \mathbb{E}\int_0^\infty \mathbf{1}_{[0,x]}(\eta_t)dt, \qquad x \ge 0.$$

For $x < 0$ we let $V(x) = 0$. For instance, if $\psi(\xi) = |\xi|^\alpha$ with $\alpha \in (0,2)$, then $V(x) = x_+^{\alpha/2}$ [15, Example 3.7]. Silverstein studied V and V' as g and ψ in [14, (1.8) and Theorem 2]. The Laplace transform of V is

$$\int_0^\infty V(x)e^{-ux}dx = \frac{1}{u\kappa(u)}, \qquad u > 0.$$

The function V is continuous and strictly increasing from $[0,\infty)$ onto $[0,\infty)$. We have $\lim_{r\to\infty} V(r) = \infty$. Also, V is subadditive:

$$V(x+y) \le V(x) + V(y), \quad x, y \in \mathbb{R}. \tag{6}$$

For a more detailed discussion of V we refer the reader to [4] and [14].

In estimates we can use V and ψ interchangeably because by [6, Lemma 1.2],

$$V(r) \approx [\psi(1/r)]^{-1/2}, \qquad r > 0. \tag{7}$$

The above means that there is a *constant*, i.e. a number $C \in (0,\infty)$, such that for all $r > 0$ we have $C^{-1}V(r) \le [\psi(1/r)]^{-1/2} \le CV(r)$. In fact in (7) we have $C = C(d)$, meaning that C may be so chosen to depend only on the dimension, see ibid. Similar notational conventions are used throughout the paper. To give full justice to V, the function is absolutely crucial in the proofs of [4], a paper leading to [6]. By (6),

$$\frac{1}{2}\varepsilon V(r) \le V(\varepsilon r) \le V(r), \qquad 0 < \varepsilon \le 1, \quad 0 < r < \infty. \tag{8}$$

2.2. **Scaling.** We shall assume relative power-type behaviors of $\psi(r)$ at infinity. Namely we say that ψ satisfies the weak lower scaling condition at infinity (WLSC) if there are numbers $\underline{\alpha} > 0$, $\underline{\theta} \in [0,\infty)$ and $\underline{C} \in (0,1]$, such that

$$\psi(\lambda r) \ge \underline{C}\lambda^{\underline{\alpha}}\psi(r) \quad \text{for} \quad \lambda \ge 1, \quad r > \underline{\theta}.$$

Put differently and more explicitly, $\psi(r)/r^{\underline{\alpha}}$ is almost increasing on $(\underline{\theta},\infty)$, i.e.

$$\frac{\psi(s)}{s^{\underline{\alpha}}} \ge \underline{C}\frac{\psi(r)}{r^{\underline{\alpha}}}, \quad \text{if} \quad s \ge r > \underline{\theta}.$$

In short we write $\psi \in \mathrm{WLSC}(\underline{\alpha},\underline{\theta},\underline{C})$, $\psi \in \mathrm{WLSC}(\underline{\alpha},\underline{\theta})$, $\psi \in \mathrm{WLSC}(\underline{\alpha})$ or $\psi \in \mathrm{WLSC}$, depending on how specific we wish to be about the constants. If $\psi \in$WLSC$(\underline{\alpha},\underline{\theta})$, then

we say that ψ satisfies the *global* weak lower scaling condition (global WLSC) if $\underline{\theta} = 0$. If $\underline{\theta} \geq 0$, then we can emphasize this by calling the scaling *local* at infinity. We always assume that $\psi \not\equiv 0$, therefore in view of $\psi \in$WLSC we have the Hartman-Wintner condition (5) satisfied, and so $\mathbb{R}^d \ni x \mapsto p_t(x)$ is smooth for each $t > 0$.

Similarly, the weak upper scaling condition at infinity (WUSC) means that there are numbers $\overline{\alpha} < 2$, $\overline{\theta} \geq 0$ and $\overline{C} \in [1, \infty)$ such that

$$\psi(\lambda r) \leq \overline{C} \lambda^{\overline{\alpha}} \psi(r) \quad \text{for} \quad \lambda \geq 1, \quad r > \overline{\theta}.$$

In short, $\psi \in \mathrm{WUSC}(\overline{\alpha}, \overline{\theta}, \overline{C})$ or $\psi \in$ WUSC. *Global* WUSC is $\mathrm{WUSC}(\overline{\alpha}, 0)$, etc.

We call $\underline{\alpha}$, $\underline{\theta}$, $\underline{C}$, $\overline{\alpha}$, $\overline{\theta}$, $\overline{C}$ the scaling characteristics of ψ. As pointed out in [6, Remark 1.4], by inflating $\underline{C}$ and $\overline{C}$ we can replace $\underline{\theta}$ with $\underline{\theta}/2$ and $\overline{\theta}$ by $\overline{\theta}/2$ in the scalings, therefore we can always choose the same, arbitrarily small value $\theta = \underline{\theta} = \overline{\theta} > 0$ in both local scalings WLSC and WUSC, if they hold at all. The scalings characterize the so-called common bounds for $p_t(x)$ [5, Theorem 21 and Theorem 26], and so they are natural conditions on ψ in the unimodal setting. The reader may also find in [5] many examples of Lévy-Khintchine exponents which satisfy WLSC or WUSC. For instance $\psi(\xi) = |\xi|^\alpha$, the Lévy-Khintchine exponent of the isotropic α-stable Lévy process in $\mathbb{R}^d$ with $\alpha \in (0, 2)$, satisfies $\mathrm{WLSC}(\alpha, 0, 1)$ and $\mathrm{WUSC}(\alpha, 0, 1)$. The characteristic exponent $\psi(\xi) = (1 + |\xi|^2)^{\alpha/2} - 1$ of the relativistic α-stable Lévy process with $\alpha \in (0, 2)$ satisfies $\mathrm{WLSC}(\alpha, 0)$ and $\mathrm{WUSC}(\alpha, 1)$. Other examples include $\psi(\xi) = |\xi|^{\alpha_1} + |\xi|^{\alpha_2} \in \mathrm{WLSC}(\alpha_1, 0, 1) \cap \mathrm{WUSC}(\alpha_2, 0, 1)$, where $0 < \alpha_1 < \alpha_2 < 2$, etc. If $\psi(r)$ is α-regularly varying at infinity and $0 < \alpha < 2$, then $\psi \in \mathrm{WLSC}(\underline{\alpha}) \cap \mathrm{WUSC}(\overline{\alpha})$, with any $0 < \underline{\alpha} < \alpha < \overline{\alpha} < 2$. The connection of the scalings to the so-called Matuszewska indices of $\psi(r)$ is explained in [5, Remark 2 and Section 4].

If $\psi \in \mathrm{WLSC}(\underline{\alpha}, \theta)$, then by (7) (or see [6, (1.8)]) we get the following scaling at 0:

$$V(\varepsilon r) \leq C \varepsilon^{\underline{\alpha}/2} V(r), \qquad 0 < \varepsilon \leq 1, \quad 0 < r < 1/\theta. \tag{9}$$

Here the range is $0 < r < \infty$ if the lower scaling of ψ is global, in agreement with (9) and the convention $1/0 = \infty$. If $\psi \in \mathrm{WUSC}(\overline{\alpha}, \theta)$, then, similarly,

$$V(\varepsilon r) \geq C \varepsilon^{\overline{\alpha}/2} V(r), \qquad 0 < \varepsilon \leq 1, \quad 0 < r < 1/\theta. \tag{10}$$

We shall need V^{-1}, the inverse function of V on $[0, \infty)$. We let

$$T(t) = V^{-1}(\sqrt{t}), \qquad t \geq 0. \tag{11}$$

Put differently, $[V(T(t))]^2 = t$. For instance, $T(t) = t^{1/\alpha}$ for the isotropic α-stable Lévy process. The functions V and T allow us to handle intrinsic difficulties which hampered extensions of [16, 1, 3, 12] to general unimodal Lévy processes, namely the lack of explicit formulas and estimates for the involved potential-theoretic objects.

We note that $T(t) < a$ if and only if $t < V^2(a)$, wherever $a, t \geq 0$. The scaling properties of T at zero reflect those of ψ (at infinity) as follows.

Lemma 2.1. *If* (9) *holds,* $0 < \varepsilon \leq 1$ *and* $0 \leq t < V(1/\theta)^2$*, then* $T(\varepsilon t) \geq c \varepsilon^{1/\underline{\alpha}} T(t)$*. If* (10) *holds,* $0 < \varepsilon \leq 1$ *and* $0 \leq t < V(1/\theta)^2$*, then* $T(\varepsilon t) \leq c \varepsilon^{1/\overline{\alpha}} T(t)$*.*

Proof. To prove the first assertion we note that T is increasing. If $0 < t < V(1/\theta)^2$, and $0 \le \varepsilon \le 1$, then $T(t) < 1/\theta$ and $T(\varepsilon t)/T(t) \le 1$. By (9),

$$\sqrt{\varepsilon} = \frac{V(T(\varepsilon t))}{V(T(t))} \le C\left(\frac{T(\varepsilon t)}{T(t)}\right)^{\underline{\alpha}/2},$$

as needed. The proof of the second inequality is analogous but uses (10). □

By (8) and the proof of Lemma 2.1 we always have

$$T(\varepsilon t) \le c\sqrt{\varepsilon}T(t), \qquad 0 < \varepsilon \le 1, \quad 0 < r < \infty. \tag{12}$$

In what follows we always assume that ν is an infinite unimodal Lévy measure on $\mathbb{R}^d$ with $d \ge 2$ and the Lévy-Khintchine exponent defined by (4) satisfies

$$\psi \in \mathrm{WLSC}(\underline{\alpha}, \theta) \cap \mathrm{WUSC}(\overline{\alpha}, \theta),$$

where $0 < \underline{\alpha} \le \overline{\alpha} < 2$, and $\theta \ge 0$. Many partial results below need less assumptions but for simplicity of presentation we leave such observations to the interested reader.

Definition. *We say that* $(\mathbf{H})$ *holds if for every* $r > 0$ *there is* $H_r \ge 1$ *such that*

$$V(z) - V(y) \le H_r V'(x)(z-y) \quad \textit{whenever} \quad 0 < x \le y \le z \le 5x \le 5r.$$

We say that $(\mathbf{H}^*)$ *holds if* $H_\infty := \sup_{r>0} H_r < \infty$*.*

We may and do chose H_r nondecreasing in r. By [4, Section 7.1], $(\mathbf{H})$ always holds in our setting because ψ satisfies WLSC and WUSC. If $\psi \in$WLSC$(\underline{\alpha}, 0)\cap$WUSC$(\overline{\alpha}, 0)$, then $(\mathbf{H}^*)$ even holds.

2.3. **Heat kernel.** By [6, Lemma 1.3], there is a $C_1 = C_1(d)$ such that

$$p_t(x) \le C_1 \frac{t}{|x|^d V^2(|x|)}, \qquad t > 0,\ x \in \mathbb{R}^d \setminus \{0\}, \tag{13}$$

hence [5, (15)],

$$\nu(x) \le C_1 \frac{1}{V^2(|x|)|x|^d}, \qquad x \ne 0. \tag{14}$$

Since $\psi \in$WLSC$(\underline{\alpha}, \theta)$, by [6, Lemma 1.5] we have

$$p_t(x) \le cT^{-d}(t), \qquad t < V^2(\theta^{-1}), \qquad x \in \mathbb{R}^d. \tag{15}$$

We now discuss the heat remainder and the heat kernel of open sets $D \subset \mathbb{R}^d$. As usual, $0 \le r_D(t,x,y) \le p_t(x-y)$. Indeed, one directly checks that $[0,t) \ni s \mapsto Y_s = p(t-s, X_s, y)$ is a $\mathbb{P}_x$-martingale for each $x, y \in \mathbb{R}^d$. The martingale almost surely converges to 0 as $s \to t$, and we let $Y_t = 0$. By optional stopping, quasi-left continuity of X and Fatou's lemma, for every stopping time $T \le t$ we have $\mathbb{E}_x Y_T \le \mathbb{E}_x Y_0 = p(t,x,y)$. The inequality $r_D(t,x,y) \le p_t(x-y)$ follows by taking $T = \tau_D \wedge t$. The next result is a consequence of the strong Markov property of X_t.

Lemma 2.2. *Consider open sets* $D \subset F \subset \mathbb{R}^d$*. For all* $t > 0$ *and* $x, y \in \mathbb{R}^d$*,*

$$p_F(t,x,y) - p_D(t,x,y) = \mathbb{E}^y\left[\tau_D < t, X(\tau_D) \in F \setminus D;\ p_F(t-\tau_D, X(\tau_D), x)\right].$$

Proof. We repeat verbatim the proof of [1, Proposition 2.3]. □

Here is a well-known Ikeda-Watanabe formula for the joint distribution of $X(\tau_D)$ and τ_D, see [10, Proposition 2.5] or [7, (27)] for proof.

Lemma 2.3. *Let $D \subset \mathbb{R}^d$ be open. For $x \in D$, $t_2 \geq t_1 \geq 0$ and $A \subset (\overline{D})^c$,*

$$\mathbb{P}^x(X(\tau_D) \in A, t_1 < \tau_D < t_2) = \int_D \int_{t_1}^{t_2} p_D(s,x,y)ds \int_A \nu(y-z)dzdy.$$

We denote $\delta_D(x) := \mathrm{dist}(x, D^c)$, $x \in \mathbb{R}^d$.

Lemma 2.4. *We have*

$$r_D(t,x,y) \leq CT(t)^{-d}, \tag{16}$$

and

$$r_D(t,x,y) \leq C_1 \frac{t}{V^2(\delta_D(x))\delta_D^d(x)}, \qquad x, y \in \mathbb{R}^d. \tag{17}$$

Proof. Since $\psi \in \mathrm{WLSC}(\underline{\alpha}, \theta, \underline{C})$, we have (15), which yields (16). By (1), (13), and symmetry,

$$r_D(t,x,y) = r_D(t,y,x) \leq \mathbb{E}^y \left[\tau_D < t;\ C_1 \frac{t - \tau_D}{V^2(|X(\tau_D) - x|)|X(\tau_D) - x|^d} \right].$$

Since $|X(\tau_D) - x| \leq \delta_D(x)$ and V is increasing, we obtain (17). □

Recall that $\mathbb{H}$ is a half-space and $C_{\mathbb{H}}(t)$ is defined immediately before Theorem 1.1.

Lemma 2.5. *If $T(t) < 1/\theta$, then $C_{\mathbb{H}}(t) \leq cT(t)^{-d+1}$.*

Proof. Denote $r(t,q) = r_{\mathbb{H}}(t,(q,0,\cdots,0),(q,0,\cdots,0))$. By (17) and (9),

$$\int_{T(t)}^\infty r(t,q)dq \leq c \int_{T(t)}^\infty \frac{V^2(T(t))}{V^2(q)q^d} dq \leq c \int_{T(t)}^\infty \frac{T(t)^{\underline{\alpha}}}{q^{d+\underline{\alpha}}} dq = cT(t)^{1-d}.$$

Using (16) we get

$$\int_0^{T(t)} r(t,q)dq \leq c \int_0^{T(t)} T(t)^{-d} dq = cT(t)^{1-d}.$$

□

To obtain a lower bound for $C_{\mathbb{H}}(t)$ we shall use the existing heat kernel estimates for geometrically regular domains. Recall that open set $D \subset \mathbb{R}^d$ satisfies the inner (outer) ball condition at scale $R > 0$ if for every $Q \in \partial D$ there is a ball $B(x', R) \subset D$ (a ball $B(x'', R) \subset D^c$) such that $Q \in \partial B(x', R)$ ($Q \in \partial B(x'', R)$, respectively). An open set D is R-smooth if it satisfies both the inner and the outer ball conditions at some scale $R > 0$. We call $B(x', R)$ and $B(x'', R)$ the inner ball and the outer ball, respectively.

In the next lemma we collect a number of results from [6]. For brevity in what follows we sometimes write $T = T(t)$, where $t > 0$ is given.

Lemma 2.6. *Let open $D \subset \mathbb{R}^d$ satisfy the outer ball condition at scale $R < 1/\theta$. There is a constant c such that for $T \vee |x-y| < 1/\theta$,*

$$p_D(t,x,y) \le c\left(\frac{V(\delta_D(x))}{V(T\wedge R)} \wedge 1\right)\left(\frac{V(\delta_D(y))}{V(T\wedge R)} \wedge 1\right)\left(T^{-d} \wedge \frac{V^2(T)}{|x-y|^d V^2(|x-y|)}\right).$$

Proof. We have (**H**). We note that $\sqrt{t} = V(T)$ and use the second part of [6, Corollary 2.4]. We need to justify that the quotient $H_R/J^4(R)$ is bounded, where H_R is the constant from (**H**) and $J(R) = \inf_{0<r\le R} \nu(B(0,r)^c)V^2(r)$. To this end we observe that H_R is increasing, and $J(R)$ is nonincreasing, hence we get an upper bound for this quotient by replacing R with $1/\theta$. If $\theta = 0$, which we also allow, then by [4, Proposition 5.2, Lemma 7.2 and 7.3] the quotient is bounded as a function of R. By [6, Lemma 1.6] with $r = 1/2$, we also have $p_{t/2}(0) \le cT^{-d}(t)$. □

Lemma 2.7. *We have $C_{\mathbb{H}}(t) \approx T(t)^{-d+1} \approx p_t(0)T(t)$ as $t \to 0$.*

Proof. By Lemma 2.6 and (2) there is $\varepsilon > 0$ such that $r(t,q) \ge \frac{1}{2}p_t(0)$ if $V(q) < \varepsilon\sqrt{t}$. Since $\psi \in$WUSC, by scaling of V there is $c > 0$ such that for $0 < q \le cT(t)$ the condition is satisfied and we have

$$\int_0^{cT(t)} r(t,q)dq \ge \frac{1}{2}\int_0^{cT(t)} T(t)^{-d}dq = \frac{c}{2}T(t)^{1-d}.$$

By WUSC and WLSC we have $p_t(0) \approx T(t)^{-d}$, see [5, (23)]. □

2.4. **Green function.** For $M \ge 0$, the truncated Green function of D is defined as

$$G_D^M(x,y) = \int_0^M p_D(t,x,y)dt, \qquad x,y \in \mathbb{R}^d.$$

The Green function of D is

$$G_D(x,y) = \int_0^\infty p_D(t,x,y)dt = G_D^\infty(x,y).$$

Lemma 2.8. *Let open $D \subset \mathbb{R}^d$ satisfy the outer ball condition at scale $R < 1/\theta$, $x,y \in \mathbb{R}^d$ and $|x-y| < 1/\theta$. Let $M = V^2(R)$. Then*

$$G_D^M(x,y) \le c\frac{V(\delta_D(y))V(\delta_D(x))}{|x-y|^d}, \tag{18}$$

and

$$G_D^M(x,y) \le c\frac{V(\delta_D(y))V(|x-y|)}{|x-y|^d}. \tag{19}$$

Furthermore, if $d > 2$ or WUSC$(\overline{\alpha},0)$ *holds, then (18) and (19) even hold for $M = V^2(1/\theta)$, including the case of global* WLSC *($M=\infty$).*

Proof. Assuming $T < R \wedge |x-y|$, by Lemma 2.6 we get

$$p_D(t,x,y) \le cV(\delta_D(y))\frac{V(T\wedge\delta_D(x))}{V^2(|x-y|)|x-y|^d},$$

hence

$$\begin{aligned}\int_0^{V^2(|x-y|\wedge R)} p_D(t,x,y)dt &\le c\frac{V(\delta_D(x))}{V^2(|x-y|)|x-y|^d}\int_0^{V^2(|x-y|\wedge R)} V(T\wedge\delta_D(x))dt\\ &\le c\frac{V(\delta_D(x))V^2(|x-y|\wedge R)V(|x-y|\wedge\delta_D(x))}{|x-y|^d V^2(|x-z|)}\\ &\le c\frac{V(\delta_D(x))V(|x-y|\wedge\delta_D(x))}{|x-y|^d}.\end{aligned}$$

This establishes (19) and (18) for small times. Then,

$$\int_{V^2(|x-y|)}^{V^2(R)} p_D(t,x,y)dt \le cV(\delta_D(x))\int_{V^2(|x-y|)}^{V^2(R)} \frac{T^{-d}(t)}{\sqrt{t}}dt.$$

By WUSC and Lemma 2.1,

$$\frac{1}{T(t)} \le \frac{c\varepsilon^{1/\overline{\alpha}}}{T(\varepsilon t)}.$$

With this in mind we obtain

$$\begin{aligned}\int_{V^2(|x-y|)}^{V^2(R)} \frac{T^{-d}(t)}{\sqrt{t}}dt &\le c\int_{V^2(|x-y|)}^{\infty} \frac{V^{2d/\overline{\alpha}}(|x-y|)}{t^{d/\overline{\alpha}+1/2}T^d(V^2(|x-y|))}dt\\ &= c\frac{V^{d/\overline{\alpha}}(|x-y|)}{|x-y|^d}\left[V^2(|x-y|)\right]^{-d/\overline{\alpha}-1/2+1},\end{aligned}$$

where the integral converges, because $d/\overline{\alpha}+1/2>1$ (recall that $\overline{\alpha}<2$). We thus get (19). To finish the proof of (18) we note that

$$\int_{V^2(|x-y|)}^{V^2(R)} p_D(t,x,y)dt \le cV(\delta_D(x))V(\delta_D(y))\int_{V^2(|x-y|)}^{V^2(R)} \frac{T^{-d}(t)}{t}dt,$$

and we proceed as before. □

2.5. **Poisson kernel.** For $M\ge 0$, the truncated Poisson kernel is defined as

$$K_D^M(x,z)=\int_D G_D^M(x,y)\nu(y-z)dy, \qquad x\in D,\ z\in D^c.$$

Lemma 2.9. *Let open $D\subset\mathbb{R}^d$ satisfy the outer ball condition at scale R. If* $\operatorname{diam}(D\cup\{z\})<1/\theta$, *then*

$$K_D^{V(R^2)/2}(x,z)\le\frac{V(\delta_D(x))}{V(\delta_D(z))}\frac{c}{|x-z|^d}, \qquad x\in D,\ z\in D^c.$$

Proof. The previous lemma gives an estimate for $G_D^{V^2(R)}$, and the Lévy measure is controlled by (14). Thus,

$$K_D^{V(R^2)/2}(x,z)\le cV(\delta_D(x))\int_D\frac{V(|x-y|)\wedge V(\delta_D(y))}{|x-y|^d|y-z|^d V^2(|y-z|)}dy.$$

Note that $|x-y| \geq |x-z|/2$ or $|y-z| \geq |x-z|/2$. Furthermore, if $|x-y| \geq |y-z|$, then $|x-y| \geq |x-z|/2$. Therefore, it is enough to verify that

$$I := \int_D \frac{V(\delta_D(y))}{|y-z|^d V^2(|y-z|)} dy \leq \frac{C}{V(\delta_D(z))}, \quad \text{and}$$
$$II := \int_{D\cap\{|x-y|<|y-z|\}} \frac{V(|x-y|)}{|x-y|^d V^2(|y-z|)} dy \leq \frac{C}{V(\delta_D(z))}.$$

Considering I we note that $\delta_D(y) \leq |y-z|$, hence

$$I \leq \int_{|y-z|>\delta_D(z)} \frac{|y-z|^{-d}}{V(|y-z|)} dy \leq c \int_{\delta_D(z)}^{1/\theta} \frac{dr}{rV(r)}$$

Using the scaling (9) we get

$$I \leq \frac{c}{V(\delta_D(z))} \int_{\delta_D(z)}^{\infty} \left(\frac{\delta_D(z)}{r}\right)^{\underline{\alpha}/2} \frac{dr}{r} = \frac{c}{V(\delta_D(z))}.$$

To verify the estimate for II we also use the scaling properties of V. For $y \in D$ we have $|y-z| < 1/\theta$, hence

$$\begin{aligned} II &\leq c \int_{|x-y|\leq|y-z|} \left(\frac{|x-y|}{|y-z|}\right)^{\underline{\alpha}/2} \frac{dy}{|x-y|^d V(|y-z|)} \\ &\leq \frac{c}{V(\delta_D(z))} \int_0^{|y-z|} \left(\frac{r}{|y-z|}\right)^{\underline{\alpha}/2} \frac{dr}{r} = \frac{c}{V(\delta_D(z))} \frac{2}{\underline{\alpha}}. \end{aligned}$$

□

3. Proof of the main result

For the convenience of the reader in the following statement we repeat our standing assumptions; see also the definition of V in Section 2.1 and that of T in (11).

Theorem 3.1. *Let ν be an infinite unimodal Lévy measure on $\mathbb{R}^d$ with $d \geq 2$, and let the Lévy-Khintchine exponent (4) satisfy $\psi \in \mathrm{WLSC}(\underline{\alpha}, \theta) \cap \mathrm{WUSC}(\overline{\alpha}, \theta)$, where $0 < \underline{\alpha} \leq \overline{\alpha} < 2$ and $\theta \geq 0$. Let open bounded set $D \subset \mathbb{R}^d$ be R-smooth with $0 < R < 1/\theta$. There is a constant c_θ depending only on ν and θ such that if $0 < t < V^2(\theta^{-1})$, or $T(t) < 1/\theta$, then the trace (3) of the Dirichlet heat kernel (2) satisfies*

$$\left| \mathrm{tr}(t, D) - |D| p_t(0) + |\partial D| C_{\mathbb{H}}(t) \right| \leq c_\theta |D| p_t(0) \frac{T(t)^2}{R^2}. \tag{20}$$

If $\theta = 0$, then (20) holds for all $t > 0$.

Recall that Lemma 2.7 asserts that $C_{\mathbb{H}}(t) \approx p_t(0)T(t)$ and $p_t(0) \approx T(t)^{-d}$ as $t \to 0$, so the approximation of the trace in Theorem 3.1 is given in terms of powers of $T(t)$.

Proof of Theorem 1.1. The result is a direct consequence of (15), Lemma 2.7 and Theorem 3.1, where we take $\theta > 0$ so small that $R < 1/\theta$ (see Section 2.2 in this connection). □

In the course of the proof of Theorem 3.1, which now follows, we usually write $T = T(t)$. As mentioned in the Introduction,

$$\mathrm{tr}(t,D) - |D|p_t(0) = \int_D p_D(t,x,x)dx - \int_D p(t,x,x)dx = -\int_D r_D(t,x,x)dx.$$

We only need to show that

$$\left|\int_D r_D(t,x,x)dx - |\partial D|C_{\mathbb{H}}(t)\right| \leq \frac{cT^2}{T^d R^2}. \tag{21}$$

We first consider $T = T(t) \geq R/2$, and we have

$$\int_D r_D(t,x,x) \leq \int_D p_t(0)dx \leq |D|p_t(0) \leq 4|D|p_t(0)\frac{T^2}{R^2}.$$

By Lemma 2.5,

$$|\partial D|C_{\mathbb{H}}(t) = |\partial D|\int_0^\infty r_{\mathbb{H}}(t,(q,0,\cdots,0),(q,0,\cdots,0))dq \leq \frac{c|D|}{R}T^{1-d} \leq \frac{c|D|T^{2-d}}{R^2}.$$

By [5, (23)], we see that (20) holds trivially in this case.

From now on we assume that $T < R/2$. For $r > 0$ we let $D_r = \{x \in D : \delta_D(x) > r\}$. We have $D = D_{R/2} \cup (D \setminus D_{R/2})$. In analyzing the decomposition we shall often use our assumptions $R < 1/\theta$ and $|x-y| < 1/\theta$, and the heat kernel estimates from Lemma 2.6. By Lemma 2.4,

$$\int_{D_{R/2}} r_D(t,x,x)dx \leq C|D_{R/2}|\frac{V^2(T)}{V^2(R/2)R^d} \leq C|D|\frac{1}{R^2R^{d-2}} \leq C|D|\frac{1}{R^2T^{d-2}}. \tag{22}$$

Thus, the integral gives insignificant contribution to the trace.

To handle the integration near ∂D, we shall estimate the heat remainder of D using the heat remainder of halfspace. Let $x^* \in \partial D$ be such that $|x - x^*| = \delta_D(x)$. Let I and O be the (inner and outer) balls with radii R such that $\partial I \cap \partial O = \{x^*\}$ and $I \subset D \subset O^c$. Let $\mathbb{H}(x)$ denote the halfspace satisfying $I \subset \mathbb{H}(x) \subset O^c$. By domain monotonicity of the heat remainder, and by Lemma 2.2,

$$\begin{aligned}|r_D(t,x,x) - r_{\mathbb{H}(x)}(t,x,x)| &\leq r_I(t,x,x) - r_{O^c}(t,x,x)\\ &= p_{O^c}(t,x,x) - p_I(t,x,x)\\ &= \mathbb{E}^x\left[\tau_I < t, X(\tau_I) \in O^c; p_{O^c}(t-\tau_I, X(\tau_I), x)\right].\end{aligned}$$

The next result is an analogue of [1, Proposition 3.1].

Proposition 3.2. *If $T < R/2$, then*

$$\mathbb{E}^x\left[\tau_I < t, X(\tau_I) \in O^c; p_{O^c}(t-\tau_I, X(\tau_I), x)\right] \leq \frac{c}{R}\left(\frac{V(T)}{\delta_D(x)^{d-1}V(\delta_D(x))} \wedge T^{1-d}\right).$$

The proof of Proposition 3.2 is given in Section 4.

Lemma 3.3. *If $T < R/2$, then*

$$\left| \int_{D \setminus D_{R/2}} r_D(t,x,x) - r_{\mathbb{H}(x)}(t,x,x)\ dx \right| \le \frac{c|D|T^2}{R^2 T^d}. \tag{23}$$

Proof. This is an analog of [1, Claim 2] and is proved as follows. By the coarea formula and Proposition 3.2 we find that the left side of (23) is bounded above by

$$\frac{cT}{RT^d} \int_0^{R/2} |\partial D_q| \left(\frac{T^{d-1}V(T)}{q^{d-1}V(q)} \wedge 1 \right) dq.$$

Therefore [1, Corollary 2.14(i)] gives a simplified bound

$$\frac{c|\partial D|}{RT^{d-1}} \int_0^{R/2} \left(\frac{T^{d-1}V(T)}{q^{d-1}V(q)} \wedge 1 \right) dq.$$

The integral over $(0,T)$ is clearly bounded by T. To estimate the integral from T to $R/2$ we note that scaling (9) for $q \in [T, R/2)$ yields $V(T) \le C(T/q)^{\underline{\alpha}/2} V(q)$. Also,

$$\int_T^{R/2} q^{1-d-\underline{\alpha}/2} dq \le \int_T^{\infty} q^{1-d-\underline{\alpha}/2} dq < \infty,$$

since $d + \underline{\alpha}/2 > 2$. $\square$

Recall that $r(t,q) = r_{\mathbb{H}}(t,(q,0,\cdots,0),(q,0,\cdots,0))$, and $C_{\mathbb{H}}(t) = \int_0^\infty r(t,q)dq$.

Lemma 3.4. *If $T < R/2$, then*

$$\left| \int_{D \setminus D_{R/2}} r_{\mathbb{H}(x)}(t,x,x)dx - |\partial D| \int_0^{R/2} r(t,q)dq \right| \le \frac{c|D|T^2}{R^2 T^d}. \tag{24}$$

Proof. Using the coarea formula we get

$$\int_{D \setminus D_{R/2}} r_{\mathbb{H}}(x)(t,x,x)dx = \int_0^{R/2} |\partial D_q| r(t,q)dq.$$

Hence the left side of the inequality (24) is bounded by

$$\int_0^{R/2} \big| |\partial D_q| - |\partial D| \big| r(t,q)dq \le \frac{C|D|}{R^2} \int_0^{R/2} q\ r(t,q)dq,$$

as follows from [1, Corollary 2.14(iii)]. For $q \in (0,T]$ we have $r(t,q) \le p_t(0)$, hence

$$\int_0^T q r(t,q)dq \le c \int_0^T \frac{q}{T^d} dq = cT^{2-d}.$$

For the remaining integration, using (17) and (9), we get

$$\begin{aligned} \int_T^{1/\theta} q r(t,q)dq &\le c \int_T^{1/\theta} \frac{t}{q^{d-1}V^2(q)} dq \le c \int_T^{1/\theta} \frac{V^2(T)}{q^{d-1}V^2(q)} dq \\ &\le c \int_T^{1/\theta} \left(\frac{T}{q}\right)^{\underline{\alpha}} \frac{dq}{q^{d-1}} \le cT^{2-d} \int_1^\infty q^{-d+1-\underline{\alpha}} dq. \end{aligned}$$

The last integral converges since $d \ge 2$ and $\underline{\alpha} > 0$. $\square$

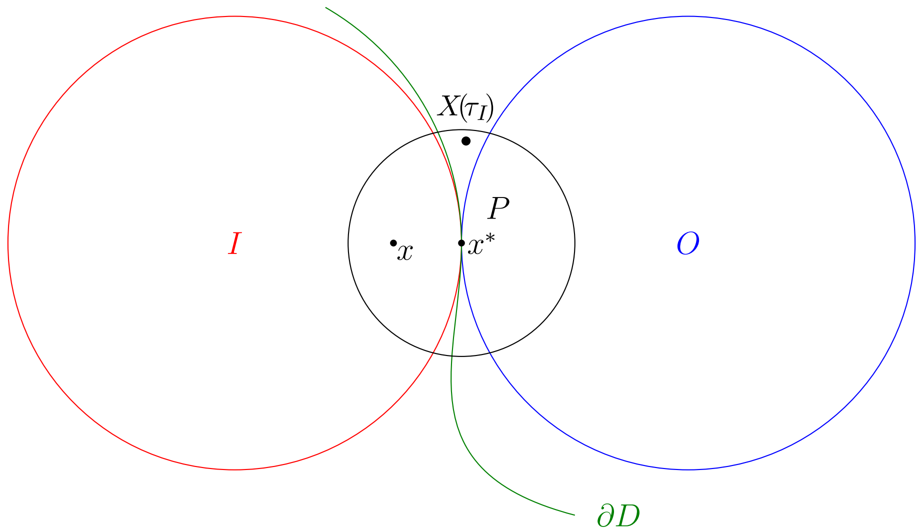

FIGURE 1. Balls $I \subset D$ (left), $O \subset D^c$ (right) and P (middle), and "a short jump" to point $X(\tau_I)$. Here $x \in P$ and $|x| = \delta_I(x)$.

Thus, for $T < R/2$ we have by Lemma 2.4

$$|\partial D| \int_{R/2}^{\infty} r(t,q)dq \le \frac{c|D|}{R} \int_{R/2}^{\infty} \frac{V^2(T)}{q^d V^2(q)} dq \le \frac{c|D|}{R} \int_{R/2}^{\infty} \frac{dq}{T^{d-2}q^2} = \frac{CT^2}{R^2T^d},$$

which is a lower order term. By Lemma 3.3, Lemma 3.4 and (22) we obtain (21).

4. Proof of Proposition 3.2.

Let $x^* = 0$, $a = (-R, 0, \dots, 0)$, $b = (R, 0, \dots, 0)$, $I = B(a, R)$ and $O = B(b, R)$. This also means that $x = (x_0, 0, \dots, 0)$ with $0 \le x_0 < R/2$, and $\delta_I(x) = |x|$, see Figure 1. Recall that $t < V^2(R/2)$ or equivalently $T < R/2$. Before we proceed to the heart of the matter we need the following lemma based on spherical integration developed in [9, Pages 355–355] and later used in [1, 2].

Lemma 4.1. *For $s < R$ we have*

$$\int_{(O^c \setminus I) \cap B(0,s)} \frac{dz}{|x-z|^\beta} \frac{V(\delta_{O^c}(z))}{V(\delta_I(z))} \le c \begin{cases} |x|^{d+1-\beta}/R & \text{if } \beta > d+1, \\ s^{d+1-\beta}/R & \text{if } \beta < d+1. \end{cases} \tag{25}$$

Proof. First we consider $V(x) = x^{\alpha/2}$ with $\alpha \in [0, 2)$. Let $z \in A = (O^c \setminus I) \cap B(0, s)$. Note that $|x - z| \ge |x|$. If $|x - z| \le 2|x|$, then $|z| \le |x - z| + |x| \le 3|x|$, which leads to the integral

$$\int_{A \cap \{|x-z| \le 2|x|\}} \frac{dz}{|x-z|^\beta} \frac{\delta^\alpha_{O^c}(z)}{\delta^\alpha_I(z)} \le \frac{1}{|x \wedge s|^\beta} \int_{A \cap \{|z| \le 3(|x| \wedge s)\}} \frac{\delta^\alpha_{O^c}(z)}{\delta^\alpha_I(z)} dz.$$

The last integral is similar to [1, (3.21)]. Using [1, (3.23) and (3.24)] we get the following upper bound

$$\frac{c}{|x\wedge s|^{\beta}}\int_0^{3(|x|\wedge s)}\frac{r^d}{R}dr=\frac{c(|x|\wedge s)^{d+1-\beta}}{R}.$$

If $|x-z|\geq 2|x|$, then $|x-z|\geq |z|/2$ and $|z|\geq |z-x|-|x|\geq |x|$. By [1, (3.24)],

$$\int_{A\cap\{|x-z|>2|x|\}}\frac{dz}{|x-z|^{\beta}}\frac{\delta^{\alpha}_{O^c}(z)}{\delta^{\alpha}_{I}(z)}\leq c\int_{A\cap\{s\geq|z|\geq|x|\}}\frac{1}{|z|^{\beta}}\frac{\delta^{\alpha}_{O^c}(z)}{\delta^{\alpha}_{I}(z)}dz\leq \frac{c}{R}\int_{|x|\wedge s}^{s}r^{d-\beta}dr.$$

If $\beta>d+1$, then the last integral is bounded by $c|x|^{d+1-\beta}$, while for $\beta<d+1$ we get the upper bound $cs^{d+1-\beta}$.

This settles (25) for $V(x)=x^{\alpha/2}$ with $\alpha\in[0,2)$. Note that the form of the right hand side of (25) does not depend on α.

Consider general $\psi\in$WUSC$(\overline{\alpha})$ and the corresponding ladder-height function V. Due to the scaling property (10) we have

$$\frac{V(\delta_{O^c}(z))}{V(\delta_I(z))}\leq c\frac{\delta^{\overline{\alpha}}_{O^c}(z)}{\delta^{\overline{\alpha}}_{I}(z)},\qquad \text{if } \delta_{O_c}(z)\geq\delta_I(z).$$

If $\delta_{O^c}(z)\leq\delta_I(z)$, then the fraction is bounded by 1, since V is monotone. Therefore, we can use the previous special case with $\alpha=\overline{\alpha}$ and $\alpha=0$ to finish the proof. $\square$

We return to the core proof of Proposition 3.2. In view of Lemma 2.3 we want to estimate

$$\begin{aligned}
&\mathbb{E}^x\left[\tau_I<t, X(\tau_I)\in O^c; p_{O^c}(t-\tau_I,X(\tau_I),x)\right]\\
&\qquad=\int_I\int_0^t p_I(s,x,y)\int_{O^c\setminus I}\nu(y-z)p_{O^c}(t-s,x,z)\,dzdsdy\\
&\qquad=I_1+I_2+I_3,
\end{aligned}$$

which splits the integration into three subregions, as specified and estimated below:

$$\begin{aligned}
I_1:&\qquad |z|>R/2,\\
I_2:&\qquad t/2<s<t \text{ AND } |x-z|<T \text{ AND } |z|\leq R/2,\\
I_3:&\qquad (s<t/2 \text{ OR } |x-z|>T) \text{ AND } |z|\leq R/2.
\end{aligned}$$

The setting, especially that of I_2, is illustrated on Figure 1.

4.1. **Long jump: integral I_1.** On I_1 we have $|z|>R/2$, hence $|x-z|\geq R/3$, thus by (13)

$$\begin{aligned}
I_1&=\int_I\int_0^t p_I(s,x,y)\int_{|z|>R/2}\nu(y-z)p(t-s,z,x)dsdzdy\\
&\leq\frac{ct}{R^dV^2(R/3)}\int_I\int_0^t p_I(s,x,y)\int_{P^c}\nu(y-z)dsdzdy\\
&=\frac{ct}{R^dV^2(R/3)}\mathbb{P}^x(\tau_I<t,|X(\tau_I)|>R/2)\leq\frac{cV^2(T)}{R^dV^2(R/2)},
\end{aligned}$$

where the last inequality follows from sublinearity (8) of V. Since $T < R/2$, we have

$$\frac{cV^2(T)}{R^d V^2(R/2)} \leq \frac{c}{R^d} \leq \frac{c}{RT^{d-1}}.$$

Since $|x| < R/2$, by monotonicity of V we get

$$\frac{cV^2(T)}{R^d V^2(R/2)} \leq \frac{cV(T)}{R^d V(R/2)} \leq \frac{cV(T)}{R|x|^{d-1}V(|x|)}.$$

4.2. **Long exit time and short jump: integral I_2.** Here we have $|x| \leq |x-z| < T$, and $|z| \leq |x-z| + |x| < 2T$. By Lemma 2.6, $t/2 < q < T$ and (12),

$$p_I(q,x,y) \leq T^{-d}\frac{V(\delta_I(y))}{V(T)}.$$

Let $S = (O^c \setminus I) \cap \{|z| < 2T\}$. We get the following upper bound,

$$\begin{aligned}
I_2 &= \int_I \int_{t/2}^t p_I(q,x,y) \int_S \nu(y-z) p_{O^c}(t-q,z,x)\,dq\,dz\,dy \\
&\leq c\int_I T(t)^{-d}\frac{V(\delta_I(y))}{V(T)} \int_S \frac{1}{|y-z|^d V^2(|y-z|)} G_{O^c}^{V^2(R/2)}(x,z)\,dz\,dy \\
&\leq \frac{cT^{-d}}{V(T)} \int_S \int_I \frac{V(\delta_I(z))}{|y-z|^d V(|y-z|)} \frac{G_{O^c}^{V^2(R/2)}(x,z)}{V(\delta_I(z))}\,dy\,dz,
\end{aligned}$$

where we use $\delta_I(y) \leq |y-z|$. Scaling (9) gives

$$I_2 \leq \frac{cT^{-d}}{V(T)} \int_S \int_{B^c(z,\delta_I(z))} \frac{\delta_I^{\underline{\alpha}/2}(z)}{|y-z|^{d+\underline{\alpha}/2}} \frac{G_{O^c}^{V^2(R/2)}(x,z)}{V(\delta_I(z))}\,dy\,dz.$$

We then rewrite the inner integral in spherical coordinates, use Green function estimate (18) and $|x| < T$,

$$\begin{aligned}
I_2 &\leq \frac{cT^{-d}}{V(T)} \int_{\delta_I(z)}^\infty \frac{\delta_I^{\underline{\alpha}/2}(z)\,dr}{r^{1+\underline{\alpha}/2}} \int_S \frac{V(|x|)V(\delta_{O^c}(z))}{|x-z|^d V(\delta_I(z))}\,dz \\
&\leq cT^{-d}\int_1^\infty \frac{dr}{r^{1+\underline{\alpha}/2}} \int_S \frac{V(\delta_{O^c}(z))}{|x-z|^d V(\delta_I(z))}\,dz = cT^{-d}\int_S \frac{V(\delta_{O^c}(z))}{|x-z|^d V(\delta_I(z))}\,dz. \qquad (26)
\end{aligned}$$

Using Lemma 4.1 with $\beta = d$ and $s = 2T$ we get

$$I_2 \leq \frac{cT^{1-d}}{R}.$$

Since $|x| < T$, we get the desired estimate from Proposition 3.2.

4.3. **Short exit time or medium jump: integral I_3.** Let $S = (O^c \setminus I) \cap \{|z| < R/2\}$. We have $|x-z| > T$ or $s < t/2$. In either case, Lemma 2.6 and sublinearity of V implies

$$p_{O^c}(t-s,x,z) \le \left(T^{-d} \wedge \frac{V^2(T)}{|x-z|^d V^2(|x-z|)} \right) \frac{V(\delta_{O_c}(z))}{V(T)}.$$

Therefore by Lemma 2.9,

$$\begin{aligned} I_3 &\le \int_I \int_0^{V^2(R/2)} p_I(s,x,y) \int_S \nu(y-z) \left(T^{-d} \wedge \frac{V^2(T)}{|x-z|^d V^2(|x-z|)} \right) \frac{V(\delta_{O_c}(z))}{V(T)} dz ds dy \\ &= c \int_S K_I^{V^2(R)}(x,z) \left(T^{-d} \wedge \frac{V^2(T)}{|x-z|^d V^2(|x-z|)} \right) \frac{V(\delta_{O_c}(z))}{V(T)} dz \\ &\le c \int_S \frac{V(|x|)}{V(\delta_I(z))} \frac{1}{|x-z|^d} \left(T^{-d} \wedge \frac{V^2(T)}{|x-z|^d V^2(|x-z|)} \right) \frac{V(\delta_{O_c}(z))}{V(T)} dz. \end{aligned}$$

If $|x-z| < T$, then we are satisfied with T^{-d} from the minimum and we note $V(|x|) < V(T)$. We arrive at (26), and finish the proof in the same way as in the previous cases.

We are left with the case $|x-z| > T$, and we have

$$I_3 \le cV(T) \int_S \frac{V(|x|)}{|x-z|^{2d} V^2(|x-z|)} \frac{V(\delta_{O_c}(z))}{V(\delta_I(z))} dz.$$

Since $\psi \in$WLSC($\underline{\alpha}$), we get

$$\begin{aligned} I_3 &\le cV(T) \int_S \frac{|x|^{\underline{\alpha}/2}}{|x-z|^{2d+\underline{\alpha}/2} V(|x-z|)} \frac{V(\delta_{O_c}(z))}{V(\delta_I(z))} dz \\ &\le \frac{cV(T)|x|^{\underline{\alpha}/2}}{(T \vee |x|)^{d-1} V(T \vee |x|)} \int_S \frac{V(\delta_{O_c}(z))}{|x-z|^{d+1+\underline{\alpha}/2} V(\delta_I(z))} dz, \end{aligned}$$

where the last inequality follows from the monotonicity of V, since $|x-z| \ge |x| \vee T$. Now we use Lemma 4.1 with $\beta = d + 1 + \underline{\alpha}/2$, to get

$$I_3 \le \frac{cV(T)}{(T \vee |x|)^{d-1} V(T \vee |x|) R}.$$

Here the right hand side is comparable with the required upper bound.

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